ISRAEL JOURNAL OF MATHEMATICS 86 (1994), 349–371

ON THE NUMBER OF NON-ISOMORPHIC SUBGRAPHS

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ABSTRACT

Let $\mathcal{K}$ be the family of graphs on ω_1 without cliques or independent subsets of size ω_1. We prove that

(a) it is consistent with CH that every $G \in \mathcal{K}$ has 2^{ω_1} many pairwise non-isomorphic subgraphs,

(b) the following proposition holds in L: $(*)$ *there is a* $G \in \mathcal{K}$ *such that for each partition* (A, B) *of* ω_1 *either* $G \cong G[A]$ *or* $G \cong G[B]$,

(c) the failure of $(*)$ is consistent with ZFC.

1. Introduction

We assume only basic knowledge of set theory — simple combinatorics for section 2, believing in $L \models \diamondsuit^+$ defined below for section 3, and finite support iterated forcing for section 4.

Answering a question of R. Jamison, H. A. Kierstead and P. J. Nyikos [5] proved that if an n-uniform hypergraph $G = \langle V, E\rangle$ is isomorphic to each of its induced subgraphs of cardinality $|V|$, then G must be either empty or complete. They raised several new problems. Some of them will be investigated in this paper. To present them we need to introduce some notions.

* The first author was supported by the United States Israel Binational Science Foundation, Publication 370.

** The second author was supported by the Hungarian National Foundation for Scientific Research grant no. 1805.

Received July 3, 1989 and in revised form December 16, 1992

An infinite graph $G = \langle V, E\rangle$is called **non-trivial** iff G contains no clique or independent subset of size $|V|$. Denote the class of all non-trivial graphs on ω_1 by $\mathcal{K}$. Let $\mathrm{I}(G)$ be the set of all isomorphism classes of induced subgraphs of $G = \langle V, E\rangle$ with size $|V|$.

H. A. Kierstead and P. J. Nyikos proved that $|\mathrm{I}(G)| \geq \omega$ for each $G \in \mathcal{K}$ and asked whether $|\mathrm{I}(G)| \geq 2^{\omega}$ or $|\mathrm{I}(G)| \geq 2^{\omega_1}$ hold or not. In [3] it was shown that (i) $|\mathrm{I}(G)| \geq 2^{\omega}$ for each $G \in \mathcal{K}$, (ii) under $\diamondsuit^+$ there exists a $G \in \mathcal{K}$ with $|\mathrm{I}(G)| = \omega_1$. In section 2 we show that if ZFC is consistent, then so is ZFC + CH + "$|\mathrm{I}(G)| = 2^{\omega_1}$ **for each** $G \in \mathcal{K}$". Given any $G \in \mathcal{K}$ we will investigate its partition tree. Applying the weak $\diamondsuit$ principle of Devlin and Shelah [2] we show that if this partition tree is a special Aronszajn tree, then $|\mathrm{I}(G)| > \omega_1$. This result completes the investigation of problem 2 of [5] for ω_1.

Consider a graph $G = \langle V, E\rangle$. We say that G is **almost smooth** if it is isomorphic to $G[W]$ whenever $W \subset V$ with $|V \setminus W| < |V|$. The graph G is called **quasi smooth** iff it is isomorphic either to $G[W]$ or to $G[V \setminus W]$ whenever $W \subset V$. H. A. Kierstead and P. J. Nyikos asked (problem 3) whether an almost smooth, non-trivial graph can exist. In [3] various models of ZFC were constructed which contain such graphs on ω_1. It was also shown that the existence of a non-trivial, quasi smooth graph on ω_1 is consistent with ZFC. But in that model CH failed. In section 3 we prove that $\diamondsuit^+$, and so V=L, too, implies the existence of such a graph.

In section 4 we construct a model of ZFC in which there is no quasi-smooth $G \in \mathcal{K}$. Our main idea is that given a $G \in \mathcal{K}$ we try to construct a partition (A_0, A_1) of ω_1 which is so bad that not only $G \ncong G[A_i]$ in the ground model but certain simple generic extensions can not add such isomorphisms to the ground model. We divide the class $\mathcal{K}$ into three subclasses and develop different methods to carry out our plan.

The question whether the existence of an almost-smooth $G \in \mathcal{K}$ can be proved in ZFC is still open.

We use the standard set-theoretical notation throughout, cf [4]. Given a graph $G = \langle V, E\rangle$we write $V(G) = V$ and $E(G) = E$. If $H \subset V(G)$ we define $G[H]$ to be $\langle H, E(G) \cap [H]^2\rangle$. Given $x \in V$ take $G(x) = \{y \in V\colon \{x, y\} \in E\}$. If G and H are graphs we write $G \cong H$ to mean that G and H are isomorphic. If $f\colon V(G) \to V(H)$ is a function we denote by $f\colon G \cong H$ the fact that f is an isomorphism between G and H.

Given a set X let $\mathrm{Bij}_p(X)$ be the set of all bijections between subsets of X. If $G = \langle V, E\rangle$is a graph take

$$\mathrm{Iso}_p(G) = \{f \in \mathrm{Bij}_p(V)\colon\ f\colon G[\mathrm{dom}(f)] \cong G[\mathrm{ran}(f)]\}\,.$$

We denote by $\mathrm{Fin}(X,Y)$ the set of all functions mapping a finite subset of X to Y.

Given a poset P and $p,q \in P$ we write $p\|_P q$ to mean that p and q are compatible in P.

The axiom $\diamondsuit^+$ claims that *there is a sequence* $\langle S_\alpha\colon \alpha < \omega_1\rangle$ *of countable sets such that for each* $X \subset \omega_1$ *we have a closed unbounded* $C \subset \omega_1$ *satisfying* $X \cap \nu \in S_\nu$ *and* $C \cap \nu \in S_\nu$ *for each* $\nu \in C$.

We denote by $\mathrm{TC}(x)$ the transitive closure of a set x. If κ is a cardinal take $H_\kappa = \{x\colon |\mathrm{TC}(x)| < \kappa\}$ and $\mathcal{H}_\kappa = \langle H_\kappa, \in\rangle$.

Let us denote by $\mathcal{D}_{\omega_1}$ the club filter on ω_1.

2. I(G) can be always large

THEOREM 2.1: *Assume that GCH holds and every Aronszajn-tree is special. Then* $|\mathrm{I}(G)| = 2^{\omega_1}$ *for each* $G \in \mathcal{K}$.

Remark: S. Shelah proved, [7, chapter V. §6,7], that the assumption of Theorem 2.1 is consistent with ZFC. ∎

During the proof we will apply the following definitions and lemmas.

LEMMA 2.2: *Assume that* $G \in \mathcal{K}$, $A \in [\omega_1]^{\omega_1}$ *and* $|\{G(x) \cap A\colon x \in \omega_1| = \omega_1$. *Then* $|\mathrm{I}(G)| = 2^{\omega_1}$.

Proof: See [3, theorem 2.1 and lemma 2.13]. ∎

Definition 2.3: Consider a graph $G = \langle \omega_1, E\rangle$.

1. For each $\nu \in \omega_1$ let us define the ordinal $\gamma_\nu \in \omega_1$ and the sequence $\langle \xi^\nu_\gamma\colon \gamma \le \gamma_\nu\rangle$ as follows: put $\xi^\nu_0 = 0$ and if $\langle \xi^\nu_\alpha\colon \alpha < \gamma\rangle$ is defined, then take

$$\xi^\nu_\gamma = \min\{\xi\colon \forall\alpha < \gamma\ \xi > \xi^\nu_\alpha \text{ and } (\{\xi^\nu_\alpha, \xi\} \in E \text{ iff } \{\xi^\nu_\alpha, \nu\} \in E)\}\,.$$

   If $\xi^\nu_\gamma = \nu$, then we put $\gamma_\nu = \gamma$.
2. Given $\nu, \mu \in \omega_1$ write $\nu \prec^G \mu$ iff $\xi^\nu_\gamma = \xi^\mu_\gamma$ for each $\gamma \le \gamma_\nu$.
3. Take $\mathcal{T}^G = \langle \omega_1, \prec^G\rangle$. $\mathcal{T}_G$ is called the **partition tree of G**. ∎

LEMMA 2.4: *If $G = \langle \omega_1, E\rangle \in \mathcal{K}$ with $|\mathrm{I}(G)| < 2^{\omega_1}$, then $\mathcal{T}^G$ is an Aronszajn tree.*

Proof: By the construction of $\mathcal{T}^G$, if $\nu, \mu \in \omega_1$, $\nu < \mu$ and $G(\nu) \cap \nu = G(\mu) \cap \nu$, then $\nu \prec^G \mu$. So the levels of $\mathcal{T}^G$ are countable by Lemma 2.2. On the other hand, $\mathcal{T}^G$ does not contain ω_1-branches, because the branches are prehomogeneous subsets and G is non-trivial. ∎

Definition 2.5:

1. Let F: $(2^\omega)^{<\omega_1} \to 2$ and $A \subset \omega_1$. We say that a function g: $\omega_1 \to 2$ is an **A-diamond for F** iff, for any $h \in (2^\omega)^{\omega_1}$, $\{\alpha \in A: F(h\lceil\alpha) = g(\alpha)\}$ is a stationary subset of ω_1.
2. $A \subset \omega_1$ is called **a small subset of** ω_1 iff for some F: $(2^\omega)^{<\omega_1} \to 2$ no function is an A-diamond for F.
3. $\mathcal{J} = \{A \subset \omega_1: A \text{ is a small subset of } \omega_1\}$. ∎

In [2] the following was proved:

THEOREM 2.6: *If $2^\omega < 2^{\omega_1}$, then $\mathcal{J}$ is a countably complete, proper, normal ideal on ω_1.*

After this preparation we are ready to prove Theorem 2.1.

Proof: Assume that $G = \langle\omega_1, E\rangle \in \mathcal{K}$ with $|\mathrm{I}(G)| < 2^{\omega_1}$ and a contradiction will be derived.

Since $2^{\omega_1} = \omega_2$, we can fix a sequence $\{G_\nu: \nu < \omega_1\}$ of graphs on ω_1 such that for each $Y \in [\omega_1]^{\omega_1}$ there is a $\nu < \omega_1$ with $G[Y] \cong G_\nu$. Write $G_\nu = \langle\omega_1, E_\nu\rangle$.

Consider the Aronszajn-tree $\mathcal{T}^G = \left\langle\omega_1, \prec^G\right\rangle$. Since every Aronszajn-tree is special and $\mathcal{I}$ is a countably complete ideal on ω_1, there is an antichain S in $\mathcal{T}^G$ with $S \notin \mathcal{J}$. Take

$$A = \left\{\alpha \in \omega_1: \exists\sigma \in S(\alpha \prec^G \sigma)\right\}.$$

Now property $(*)$ below holds:

$$(*) \quad \forall\sigma \in S\ \forall\rho \in (S \cup A) \setminus \sigma + 1 \exists\alpha \in A \cap \sigma\ (\{\sigma, \alpha\} \in E \text{ iff } \{\rho, \alpha\} \notin E).$$

Indeed, if for each $\alpha \in A \cap \sigma$ we had $\{\sigma, \alpha\} \in E$ iff $\{\rho, \alpha\} \in E$, then $\sigma \prec^G \rho$ would hold by the construction of $\mathcal{T}^G$.

Let $\nu \in \omega_1$, $\sigma \in S$, $T \subset S \cap \sigma$ and f: $G[(A \cap \sigma) \cup T] \to G_\nu$ be an embedding. Define $F(\nu, \sigma, T, f) \in 2$ as follows:

$$F(\nu, \sigma, T, f) = 1 \quad \text{iff } \exists x \in G_\nu(\forall\alpha \in A \cap \sigma) \quad \left(\{x, f(\alpha)\} \in E_\nu \quad \text{iff } \{\sigma, \alpha\} \in E\right).$$

In case $\omega\sigma = \sigma$, under suitable encoding, F can be viewed as a function from $(2^\omega)^{<\omega_1}$ to 2.

Since $S \notin \mathcal{J}$, there is a $g \in 2^{\omega_1}$ such that for every $\nu \in \omega_1 = 2^\omega$, $T \subset S$ and $f\colon G[A \cup T] \cong G_\nu$, the set

$$S_T = \{\sigma \in S\colon g(\sigma) = F(\nu, \sigma, T \cap \sigma, f\lceil\sigma)\}$$

is stationary. Take $T = \{\sigma \in S\colon g(\sigma) = 0\}$. Choose an ordinal $\nu < \omega_1$ and a function f with $f\colon G[A \cup T] \cong G_\nu$. For each $\sigma < \omega_1$ with $\sigma = \omega\sigma$ it follows, by (*), that

$$\sigma \in T \quad \text{iff } \exists x \in \omega_1\ \forall \alpha \in S \cap \sigma \quad (\{x, f(\alpha)\} \in E_\nu \quad \text{iff } \{\sigma, \alpha\} \in E).$$

Thus $g(\sigma) = 0$ iff $F(\nu, \sigma, T \cap \sigma, f\lceil\sigma) = 1$, for each $\sigma \in S$, that is, $S_T = \emptyset$, which is a contradiction. ∎

3. A quasi-smooth graph under $\diamondsuit^+$

THEOREM 3.1: *If $\diamondsuit^+$ holds, then there exists a non-trivial, quasi-smooth graph on ω_1.*

Proof: Given a set X, $\mathcal{A} \subset \mathrm{P}(X)$ and $\mathcal{F} \subset \mathrm{Bij}_p(X)$ take

$$\mathrm{Cl}(\mathcal{A}, \mathcal{F}) = \bigcap \big\{\mathcal{B}\colon \mathcal{B} \supset \mathcal{A} \text{ and } \forall B_0, B_1 \in \mathcal{B}\ \forall f \in \mathcal{F}\ \forall Y \in [X]^{<\omega} \\ \{B_0 \cup B_1, f''B_0, B_0 \triangle Y\} \subset \mathcal{B}\big\}.$$

We say that $\mathcal{A}$ is *$\mathcal{F}$-closed* if $\mathcal{A} = \mathrm{Cl}(\mathcal{A}, \mathcal{F})$. Given $\mathcal{A}, \mathcal{D} \subset P(X)$, we say that $\mathcal{D}$ is *uncovered by* $\mathcal{A}$ if $|D \setminus A| = \omega$ for each $A \in \mathcal{A}$ and $D \in \mathcal{D}$.

LEMMA 3.2: *Assume that $\mathcal{F} \subset \mathrm{Bij}_p(X)$ is a countable set, $\mathcal{A}^0$, $\mathcal{A}^1 \subset P(X)$ are countable, $\mathcal{F}$-closed families. If $\mathcal{D} \subset P(X)$ is a countable family which is uncovered by $\mathcal{A}^0 \cup \mathcal{A}^1$, then there is a partition (B_0, B_1) of X such that $\mathcal{D}$ is uncovered by $\mathrm{Cl}(\mathcal{A}^i \cup \{B_i\}, \mathcal{F})$ for $i < 2$.*

Proof: We can assume that $\mathcal{F}$ is closed under composition. Fix an enumeration $\{\langle D_n, k_n, F_n, i_n, A_n\rangle : n \in \omega\}$ of $\mathcal{D} \times \omega \times \mathcal{F}^{<\omega} \times \{\langle i, A\rangle : i \in 2, A \in \mathcal{A}^i\}$. By induction on n, we will pick points $x_n \in X$ and will define finite sets, B_n^0 and B_n^1, such that $B_n^0 \cap B_n^1 = \emptyset$ and $B_n^i \subset B_{n+1}^i$.

Assume that we have done it for $n-1$. Write $F_n = \langle f_0, \ldots, f_{k-1} \rangle$. Take $B_{n-1} = B^0_{n-1} \cup B^1_{n-1}$ and

$$B^-_n = B_{n-1} \cup \bigcup \left\{ f_j'' B_{n-1}: j < k \right\}.$$

Pick an arbitrary point $x_n \in D_n \setminus (A_n \cup B^-_n)$. Put

$$B^{i_n}_n = B^{i_n}_{n-1}$$

and

$$B^{1-i_n}_n = B^{1-i_n}_{n-1} \cup \{x_n\} \cup \left\{ f_j^{-1}(x_n): j < k \right\}.$$

Next choose a partition (B^0, B^1) of X with $B^i \supset \cup\{B^i_n: n < \omega\}$ for $i < 2$. We claim that it works. Indeed, a typical element of $\mathrm{Cl}(\mathcal{A}^i \cup \{B^i\}, \mathcal{F})$ has the form

$$C = A \cup \bigcup \left\{ f_j'' B^i: j < k \right\},$$

where $A \in \mathcal{A}$, $k < \omega$ and $f_0, \ldots, f_{k-1} \in \mathcal{F}$. So, if $D \in \mathcal{D}$, then

$$D \setminus C \supset \{x_n: D_n = D, A_n = A, i_n = i \text{ and } F_n = \langle f_0, \ldots, f_{k-1} \rangle\}$$

because $x_n \notin A$ and $f_j^{-1}(x_n) \in B^{1-i}$ by the constuction. ∎

Consider a sequence $F = \langle f_0, \ldots, f_{n-1} \rangle$. Given a family $\mathcal{F} \subset \mathrm{Bij}_p(X)$ we say that F is an *$\mathcal{F}$-term* provided $f_i = f$ or $f_i = f^{-1}$ for some $f \in \mathcal{F}$, for each $i < n$. We denote the function $f_0 \circ \cdots \circ f_{n-1}$ by F as well. We will assume that the empty term denotes the identity function on X. If $l \le n$ take ${}_{(l)}F = \langle f_0, \ldots, f_{l-1} \rangle$ and $F_{(l)} = \langle f_l, \ldots, f_{n-1} \rangle$. Let

$$\mathrm{Sub}(F) = \left\{ \langle f_{i_0}, \ldots, f_{i_{l-1}} \rangle : l \le n, i_0 < \cdots < i_{l-1} < n \right\}.$$

Given $f \in \mathcal{F}$ and $x, y \in X$ with $x \notin \mathrm{dom}(f)$ and $y \notin \mathrm{ran}(f)$ let $F^{f,x,y}$ be the term that we obtain replacing each occurrence of f and of f^{-1} in F with $f \cup \{\langle x, y \rangle\}$ and with $f^{-1} \cup \{\langle y, x \rangle\}$, respectively.

LEMMA 3.3: *Assume that $\mathcal{F} \subset \mathrm{Bij}_p(X)$, $\mathcal{A} \subset P(X)$ is $\mathcal{F}$-closed, $F_0, \ldots, F_{n-1}$ are $\mathcal{F}$-terms, $z_0, \ldots, z_{n-1} \in X$, $A_0, \ldots, A_{n-1} \in \mathcal{A}$ such that for each $i < n$*

$$(*) \qquad z_i \notin \bigcup \{F'' A_i: F \in \mathrm{Sub}(F_i)\}.$$

If $f \in \mathcal{F}$, $x \in X \setminus \mathrm{dom}(f)$, $Y \in [X \setminus \mathrm{ran}(f)]^\omega$ with $|A \cap Y| < \omega$ for each $A \in \mathcal{A}$, then there are infinitely many $y \in Y$ such that $(*)$ remains true when replacing f with $f \cup \{\langle x, y\rangle\}$, that is,

$$(**) \qquad z_i \notin \bigcup \left\{ F''A_i \colon F \in \mathrm{Sub}(F_i^{f,x,y}) \right\}$$

for each $i < n$.

Proof: It is enough to prove it for $n = 1$. Write $F = \langle f_0 \dots, f_{k-1}\rangle$, $A = A_0$, $z = z_0$. Take

$$Y_{F,A} = \{y \in Y \colon (**) \text{ holds for } y\}.$$

Now we prove the lemma by induction on k.

If $k = 0$, then $Y_{F,A} = Y \setminus A$. Suppose we know the lemma for $k-1$. Using the induction hypothesis we can assume that (†) below holds:

$$(\dagger) \qquad Y = \bigcap \left\{ Y_{G, F''_{(l)}A} \colon l \le n, G \in \mathrm{Sub}({}_{(l)}F^{f,x,y}), G \ne F^{f,x,y} \right\}.$$

Assume that $|Y_{F,A}| < \omega$ and a contradiction will be derived.

First let us remark that either $f_{k-1} = f$ or $f_{k-1} = f^{-1}$ by (†).

CASE 1: $f_{k-1} = f^{-1}$. Then $Y_{F,A} \supset Y \setminus A$ by (†), so we are done.

CASE 2: $f_{k-1} = f$. In this case $x \in A$ and for all but finitely many $y \in Y$ we have $z = F^{f,x,y}(x)$. Then for each $y, y' \in Y$ take

$$l(y, y') = \max \left\{ l \le n \colon \forall i < l \; F_{(i)}^{f,x,y}(x) = F_{(i)}^{f,x,y'}(x) \right\}.$$

By Ramsey's theorem, we can assume that $l(y, y') = l$ whenever $y, y' \in Y$. Clearly $l < n$. Then $F_{(l)}^{f,x,y}(x) \ne F_{(l)}^{f,x,y'}(x)$ but $F_{(l-1)}^{f,x,y}(x) = F_{(l-1)}^{f,x,y'}(x)$, so $f_l = f^{-1}$ and $F_{(l-1)}^{f,x,y}(x) = x$ for each $y \in Y$. Thus $z = {}_{(l-1)}F^{f,x,y}(x)$ for each $y \in Y$, which contradicts (†) because $x \in A$.

The lemma is proved. ∎

We are ready to construct our desired graph.

First fix a sequence $\langle M_\alpha \colon \alpha < \omega_1 \rangle$ of countable, elementary submodels of some $\mathcal{H}_\lambda$ with $\langle M_\gamma \colon \gamma < \alpha \rangle \in M_\alpha$ for each $\alpha < \omega_1$, where λ is a large enough regular cardinal.

Then choose a $\diamondsuit$-sequence $\langle S_\alpha \colon \alpha < \omega_1 \rangle \in M_0$ for the uncountable subsets of ω_1, that is , $\{\alpha < \omega_1 \colon X \cap \alpha = S_\alpha\} \notin NS(\omega_1)$ whenever $X \in [\omega_1]^{\omega_1}$. We can also assume that S_α is cofinal in α for each limit α.

We will define, by induction on α,

1. graphs $G_\alpha = \langle \omega\alpha, E_\alpha \rangle$ with $G_\beta = G_\alpha[\omega\beta]$ for $\beta < \alpha$,
2. countable sets $\mathcal{F}_\alpha \in \mathrm{Iso}_p(G_\alpha)$,

satisfying the induction hypotheses (I)–(II) below:

(I) $\{S_{\omega\gamma} \colon \gamma \le \alpha\}$ is uncovered by $I_\alpha \cup J_\alpha$ where

$$I_\alpha = \mathrm{Cl}(\{G(\nu) \cap \nu \colon \nu \in \omega\alpha\}, \bigcup_{\beta \le \alpha} \mathcal{F}_\beta)$$

and

$$J_\alpha = \mathrm{Cl}(\{\nu \setminus G(\nu) \colon \nu \in \omega\alpha\}, \bigcup_{\beta \le \alpha} \mathcal{F}_\beta).$$

To formulate (II) we need the following definition.

Definition 3.4: Assume that $\alpha = \beta + 1$ and $Y \subset \omega\alpha$. We say that Y is **large** if $\forall n \in \omega$, $\forall \langle \langle f_i, x_i \rangle : i < n \rangle$, $\forall h$

if

1. $\forall i < n \ \exists \alpha_i < \beta \ f_i \in \mathcal{F}_{\alpha_i}$,
2. $\forall i < n \ \omega\alpha_i \le x_i < \omega\beta$,
3. $\forall i < n \ \mathrm{ran}(f_i) \subset Y$,
4. $\forall i \ne j < n \ \mathrm{ran}(f_i) \cap \mathrm{ran}(f_j) = \emptyset$
5. $h \in \mathrm{Fin}(Y \cap \omega\beta, 2)$ and $\mathrm{dom}(h) \cap \bigcup \{\mathrm{ran}(f_i) \colon i < n\} = \emptyset$,

then

$\exists y \in Y \cap [\omega\beta, \omega\alpha)$ such that

6. $\forall i < n \ \forall x \in \mathrm{dom}(f_i)$ $(\{y, f_i(x)\} \in E_\alpha$ iff $\{x_i, x\} \in E_\alpha)$,
7. $\forall z \in \mathrm{dom}(h)$ $\{y, z\} \in E_\alpha$ iff $h(z) = 1$. ∎

Take

(II) If $\alpha = \beta + 1$, then $\omega\alpha$ is large. The construction will be carried out in such a way that

$\langle G_\beta \colon \beta \le \alpha \rangle \in M_\alpha$ and $\langle \mathcal{F}_\beta \colon \beta < \alpha \rangle \in M_\alpha$.

To start with take $G_0 = \langle \emptyset, \emptyset \rangle$ and $\mathcal{F} = \{\emptyset\}$. Assume that the construction is done for $\beta < \alpha$.

CASE 1: *α is limit.* We must take $G_\alpha = \cup \{G_\beta \colon \beta < \alpha\}$. We will define sets $\mathcal{F}^0_\alpha, \mathcal{F}^1_\alpha \subset \mathrm{Iso}_p(G_\alpha)$ and will take $\mathcal{F}_\alpha = \mathcal{F}^0_\alpha \cup \mathcal{F}^1_\alpha$.

Let

$$\mathcal{F}^0_\alpha = \{ f \in \mathrm{Iso}_p(G_\alpha) \cap M_\alpha \colon \exists \langle \alpha_n \colon n < \omega \rangle \subset \alpha \ \sup \{\alpha_n \colon n < \omega\} = \alpha,$$
$$f \lceil \omega\alpha_n \in \mathcal{F}_{\alpha_n} \text{ and } f \lceil \omega\alpha_n \colon G_{\alpha_n} \cong G_{\alpha_n}[\mathrm{ran}(f)] \text{ for each } n \in \omega \}.$$

Take $\mathcal{F}^- = \bigcup_{\beta<\alpha} \mathcal{F}^\beta \cup \mathcal{F}^0_\alpha$, $I^-_\alpha = \bigcup_{\beta<\alpha} I_\beta$ and $J^-_\alpha = \bigcup_{\beta<\alpha} J_\beta$. Clearly $\mathcal{F}^- \subset M_\alpha$ with $\mathcal{F}^- \in M_{\alpha+1}$, so $M_{\alpha+1} \models$ "$|\mathcal{F}^-| = \omega$". Obviously both I^-_α and J^-_α are $\mathcal{F}^-$-closed and $\mathcal{S} = \{S_{\omega\beta}: \beta \leq \alpha\}$ is uncovered by them.

From now on we work in $M_{\alpha+1}$ to construct $\mathcal{F}^1_\alpha$. For $W \subset \omega\alpha$ write $L_W = \{\nu < \alpha: W \cap (\omega\nu + \omega) \text{ is large}\}$.

Take

$$\mathcal{W}_\alpha = \Big\{ \langle W, f\rangle \in (P(\omega\alpha) \cap \mathcal{M}_\alpha) \times (\bigcup_{\beta<\alpha} \mathcal{F}_\beta): L_W \text{ is cofinal in } \alpha$$
$$\text{and } f: G_{\gamma_f} \cong G_{\gamma_f}[W \cap \omega\gamma_f] \text{ for some } \gamma_f < \alpha\Big\}.$$

We want to find functions $g^{W,f} \supset f$ for $\langle W, f\rangle \in \mathcal{W}_\alpha$ such that

(A) $g^{W,f}: G_\alpha \cong G_\alpha[W]$

(B) taking $\mathcal{F}^1_\alpha = \{g^{W,f}: \langle W, f\rangle \in \mathcal{W}_\alpha\}$ the induction hypothesis (I) remains true.

First we prove a lemma:

Lemma 3.5: *If $\langle W, f\rangle \in \mathcal{W}_\alpha$, $g \in \mathrm{Iso}_p(G_\alpha, G_\alpha[W])$, $g \supset f$, $|g \setminus f| < \omega$, then*

(i) *for each $x \in W \setminus \mathrm{dom}(f)$ the set*

$$\{y \in W: g \cup \{\langle x, y\rangle\} \in \mathrm{Iso}_p(G_\alpha, G_\alpha[W])\}$$

is cofinal in $\omega\alpha$.

(ii) *for each $y \in W \setminus \mathrm{ran}(f)$ the set*

$$\{x \in W: g \cup \{\langle x, y\rangle\} \in \mathrm{Iso}_p(G_\alpha, G_\alpha[W])\}$$

is cofinal in $\omega\alpha$.

Proof: (i) Define the function $h: \mathrm{ran}(g) \setminus \mathrm{ran}(f) \to 2$ with $h(g(z)) = 1$ iff $\{z, x\} \in E_\alpha$. Choose $\beta \in L_W$ with $\mathrm{ran}(h) \subset \omega\beta$ and $\gamma_f \leq \beta$. Since $W \cap (\omega\beta + \omega)$ is large, we have a $y \in W \cap [\omega\beta, \omega\beta + \omega)$ such that

1. $\{y, f(z)\} \in E_\alpha$ iff $\{x, z\} \in E_\alpha$ for each $z \in \mathrm{dom}(f)$,
2. $\{y, g(z)\} \in E_\alpha$ iff $h(g(z)) = i$ for each $z \in \mathrm{dom}(g) \setminus \mathrm{dom}(f)$.

But this means that $g \cup \{\langle x, y\rangle\} \in \mathrm{Iso}_p(G_\alpha, G_\alpha[W])$.

(ii) The same proof works using that $\omega\beta + \omega$ is large for each $\beta < \alpha$. ∎

By induction on n, we will pick points $z_n \in \omega\alpha$ and will construct families of partial automorphisms, $\{g^{W,f}_n: \langle W, f\rangle \in \mathcal{W}_\alpha\}$ such that $g^{W,f} = \cup \{g^{W,f}_n: n < \omega\}$ will work.

During the inductive construction we will speak about $\mathcal{F}_\alpha$-terms and about functions which are represented by them in the n^{th} step.

If $F = \langle h_0, \dots h_{k-1}\rangle$ is an $\mathcal{F}_\alpha$-term and $n \in \omega$ take $F_{[n]} = j_0 \circ \cdots \circ j_{k-1}$ where

$$j_i = \begin{cases} g_n^{W,f} & \text{if } h_i = g^{W,f}, \\ (g_n^{W,f})^{-1} & \text{if } h_i = (g^{W,f})^{-1}, \\ h_i & \text{otherwise.} \end{cases}$$

First fix an enumeration $\{\langle\langle W_n, f_n\rangle, u_n, i_n\rangle : 1 \le n < \omega\}$ of $\mathcal{W}_\alpha \times \omega\alpha \times 2$ and an enumeration $\langle\langle\langle F_{n,i}: i < l_n\rangle, j_n, \langle A_{n,i}: i < l_n\rangle, D_n\rangle\, n < \omega\rangle$ of the quadruples $\langle\langle F_0, \dots, F_{k-1}\rangle, j, \langle A_0, \dots, A_{k-1}\rangle, D\rangle$ where $k < \omega$, $F_0, \dots, F_{k-1}$ are $\mathcal{F}_\alpha$-terms, $j \in 2$, $D \in \mathcal{S}$ and either $j = 0$ and $A_0, \dots, A_{k-1} \in I_\alpha^-$ or $j = 1$ and $A_0, \dots, A_{k-1} \in J_\alpha^-$.

During the inductive construction conditions (i)–(v) below will be satisfied:

(i) $g_0^{W,f} = f$,

(ii) $g_n^{W,f} \in \mathrm{Iso}_p(G_\alpha, G_\alpha[W])$,

(iii) $g_n^{W,f} \supset g_{n-1}^{W,f}$, $|g_n^{W,f} \setminus f| < \omega$,

(iv) $z_k \notin \bigcup \left\{F''_{[n]} A_{k,i}: F \in \mathrm{Sub}(F_{k,i})\right\}$ for each $i < l_k$ and $k < n$,

(v) if $i_n = 1$, then $u_n \in \mathrm{dom}(g_n^{W_n,f_n})$,
if $i_n = 0$, then either $u_n \notin W_n$ or $u_n \in \mathrm{ran}(g_n^{W_n,f_n})$.

If $n = 0$, then take $g_0^{W,f} = f$.

If $n > 0$, then let $g_n^{W,f} = g_{n-1}^{W,f}$ whenever $\langle W, f\rangle \ne \langle W_n, f_n\rangle$. Assume that $i_n = 0$, $\langle W, f\rangle = \langle W_n, f_n\rangle$ and $u_n \notin \mathrm{dom}(g_{n-1}^{W_n,f_n})$. Then, by Lemma 3.5, the set $Y = \left\{y \in W: g_{n-1}^{W,f} \cup \{\langle u_n, y\rangle\} \in \mathrm{Iso}_p(G_\alpha, G_\alpha[W])\right\}$ is unbounded in $\omega\alpha$. Since the members of $I_\alpha^- \cup J_\alpha^-$ are bounded in $\omega\alpha$, we can apply Lemma 3.3 to pick a point $y \in Y$ such that taking $g_n^{W_n,f_n} = g_{n-1}^{W_n,f_n} \cup \{\langle u_n, y\rangle\}$ condition (iv) holds.

If $i_n = 1$ and $\langle W, f\rangle = \langle W_n, f_n\rangle$, then the same argument works.

Finally pick a point

$$z_n \notin D_n \setminus \bigcup \left\{F''_{[n]} A_{n,i}: F \in \mathrm{Sub}(F_{n,i}), i < l_n\right\}.$$

The inductive construction is done.

Take $g^{W,f} = \cup \{g_n^{W,f}: n < \omega\}$. By (v),

$$g_n^{W,f}: G_{\omega\alpha} \cong G_{\omega\alpha}[W].$$

By (iv), we have

$$z_k \in D_k \setminus \bigcup \{F''_{k,i} A_{k,i}: i < l_k\}$$

and so it follows that $\{S_{\omega\beta}: \beta \le \alpha\}$ is uncovered by $I_\alpha \cup J_\alpha$.

CASE 2: $\alpha = \beta + 1$.

To start with we fix an enumeration $\{\langle\langle\langle f_i^k, x_i^k\rangle : i < n_k\rangle, h_k\rangle : k \in \omega\}$ of pairs $\langle\langle\langle f_i, x_i\rangle : i < n\rangle, h\rangle$ satisfying 3.4.1–5.

If $k \in \omega$ take

$$B_k^0 = h_k^{-1}\{0\} \cup \{f_i^k(\nu): i < n_k,\ \nu \in \mathrm{dom}(f_i^k) \text{ and } \{\nu, x_i^k\} \notin E_\beta\}$$

and

$$B_k^1 = h_k^{-1}\{1\} \cup \{f_i^k(\nu): i < n_k,\ \nu \in \mathrm{dom}(f_i^k) \text{ and } \{\nu, x_i^k\} \in E_\beta\}.$$

Applying Lemma 3.2 ω-many times we can find partitions (C_k^0, C_k^1), $k < \omega$, of $\omega\beta$ such that, taking

$$I_\beta^+ = \mathrm{Cl}\big((I_\beta \cup \{C_k^1: k \in \omega\}, \bigcup_{\gamma\le\beta} \mathcal{F}_\gamma\big)$$

and

$$J_\beta^+ = \mathrm{Cl}\big((I_\beta \cup \{C_k^0: k \in \omega\}, \bigcup_{\gamma\le\beta} \mathcal{F}_\gamma\big),$$

the set $\{S_{\omega\gamma}: \gamma \le \beta\}$ is uncovered by $I_\beta^+ \cup J_\beta^+$.

We can assume that $B_k^i \subset C_k^i$ for $i < 2$ and $k < \omega$ because $B_k^0 \in J_\beta$ and $B_k^1 \in I_\beta$. Take

$$E_\alpha = E_\beta \cup \{\{\nu, \omega\beta + n\} : \nu < \omega\beta,\ n \in \omega \text{ and } \nu \in B_n^1\}$$

and

$$\mathcal{F}_\alpha = \emptyset.$$

By the construction of $G_\alpha = \langle\omega\alpha, E_\alpha\rangle$, it follows that $\omega\alpha$ is large, so (II) holds. On the other hand

$$I_\alpha = \{X \cup Y: X \in I_\beta^+,\ Y \in [\omega\alpha]^{<\omega}\}$$

and

$$J_\alpha = \{X \cup Y: X \in J_\beta^+,\ Y \in [\omega\alpha]^{<\omega}\},$$

so $\{S_{\omega\gamma}: \gamma < \alpha\}$ is uncovered by $I_\alpha \cup J_\alpha$. Finally $S_{\omega\alpha}$ is cofinal in $\omega\alpha$ but the elements of $I_\alpha \cup J_\alpha$ are all bounded, so the induction hypothesis (I) also holds.

The construction is done. Take $E = \bigcup\{E_\alpha: \alpha < \omega_1\}$ and $G = \langle\omega_1, E\rangle$. By (I), G is non-trivial. Finally, we must prove that G is quasi smooth. Consider a set $Y \subset \omega_1$. The following lemma is almost trivial.

LEMMA 3.6: *For each $\alpha < \omega_1$ either $Y \cap (\omega\alpha + \omega)$ or $(\omega\alpha + \omega) \setminus Y$ is large.*

Proof: Assume on the contrary that there are pairs $\langle\langle\langle f_i, x_i\rangle : i < n\rangle, h\rangle$ and $\langle\langle\langle f_i, x_i\rangle : n \le i < n+k\rangle, h'\rangle$ showing that neither $Y \cap (\omega\alpha+\omega)$ nor $(\omega\alpha+\omega)\setminus Y$ is large. Then $\langle\langle\langle f_i, x_i\rangle : i < n+k\rangle, h \cup h'\rangle$ shows that $\omega\alpha+\omega$ is not large. ∎

So we can assume that the set

$$L = \{\alpha < \omega_1 : Y \cap (\omega\alpha + \omega) \text{ is large}\}$$

is uncountable and to complete the proof of Theorem 3.1 it is enough to show that in this case $G \cong G[Y]$. By $\diamondsuit^+$, we can find a club subset $C \subset L'$ such that $Y \cap \omega\gamma \in M_\gamma$, $C \cap \omega\gamma \in M_\gamma$ and $\omega\gamma = \gamma$ whenever $\gamma \in C$. We can assume that $0 \in C$.

Write $C = \{\gamma_\nu : \nu < \omega_1\}$. By induction on $\nu < \omega_1$, we will construct functions f_ν such that

(a) $f_\nu : G_{\gamma_\nu} \cong G_{\gamma_\nu}[Y]$, $f_\nu \in \mathcal{F}_{\gamma_\nu}$,

(b) $\langle f_\mu : \mu < \nu\rangle \in M_{\sup\{\gamma_\mu+1:\, \mu<\nu\}}$.

Take $f_0 = \emptyset$. If $\nu = \mu + 1$, then let $f_\nu = g^{Y\cap\omega\gamma_\nu, f_\mu}$. If ν is limit, then put $f_\nu = \cup\{f_\mu : \mu < \nu\}$. Clearly (a) and (b) remains valid. Finally put $f = \cup\{f_\nu : \nu < \omega_1\}$. Then $f : G \cong G[Y]$, so the theorem is proved. ∎

4. A model without quasi-smooth graphs

Given an Aronszajn-tree $T = \langle\omega_1, \prec\rangle$ define the poset Q_T as follows: the underlying set of Q_T consists of all functions f mapping a finite subset of ω_1 to ω such that $f^{-1}\{n\}$ is antichain in T for each $n \in \omega$. The ordering on Q_T is as expected: $f \le_{Q_T} g$ iff $f \supset g$. For $\gamma < \omega_1$ denote by T_γ the set of elements of T with height γ. Take $T_{<\delta} = \bigcup_{\gamma<\delta} T_\gamma$. If $x \in T_\delta$ and $\gamma < \delta$, let $x \lceil \gamma$ be the unique element of T_γ which is comparable with x. We write $\mathcal{C}$ for the poset $\langle \mathrm{Fin}(\omega_1, 2), \supset\rangle$, that is, forcing with $\mathcal{C}$ adds ω_1-many Cohen reals to the ground model.

THEOREM 4.1: *If ZF is consistent, then so is ZFC + "there are no non-trivial quasi-smooth graphs on ω_1".*

Proof: Assume that GCH holds in the ground model. Consider a finite support iteration $\langle P_i, Q_j : i \le \omega_2, j < \omega_2\rangle$ satisfying (a)–(c) below:

(a) If $j < \omega_2$ is even, then $Q_j = \mathcal{C}$.

(b) If $j < \omega_2$ is odd, then $V^{P_j} \models$ "$Q_j = Q_{T_j}$ *for some Aronszajn-tree* T_j".

(c) $V^{P_{\omega_2}} \models$ "*every Aronszajn tree is special*".

We will show that $V^{P_{\omega_2}}$ does not contain non-trivial, quasi-smooth graphs on ω_1.

To start with we introduce some notation. Consider a graph $G = \langle V, E\rangle$. For $x \in V$ define the function $\mathrm{tp}_G(x)\colon V \setminus \{x\} \to 2$ by the equation $G(x) = \mathrm{tp}_G(x)^{-1}\{1\}$. Given $A \subset V$ write $\mathrm{tp}_G(x, A) = \mathrm{tp}_G(x)\lceil A$.

If $A \subset V$ and $t \in 2^A$, take $\mathrm{rl}_G(t) = \{x \in V \setminus A\colon \mathrm{tp}_G(x, A) = t\}$ and $\mathrm{rl}^*_G(t) = \{x \in V \setminus A\colon |\mathrm{tp}_G(x, A)\Delta t| < \omega\}$. For $x \in V$ and $A \subset V$ put $\mathrm{twin}_G(x, A) = \mathrm{rl}_G(\mathrm{tp}_G(x, A)) = \{y \in V \setminus A\colon \mathrm{tp}_G(x, A) = \mathrm{tp}_G(y, A)\}$.

For $A \subset V$ define the equivalence relation $\equiv_{G,A}$ on $V \setminus A$ as follows:

$$x \equiv_{G,A} y \qquad \text{iff } |\mathrm{tp}_G(x, A)\Delta \mathrm{tp}_G(y, A)| < \omega.$$

For $x \in V \setminus A$ denote by $[x]_{G,A}$ the equivalence class of x in $\equiv_{G,A}$. Clearly $[x]_{G,A} = \mathrm{rl}^*_G(\mathrm{tp}_G(x, A))$. Write $G/\equiv_{G,A}$ for the family of equivalence classes of $\equiv_{G,A}$.

We divide $\mathcal{K}$ into three subclasses, $\mathcal{K}_0$, $\mathcal{K}_1$ and $\mathcal{K}_2$, and investigate them separately to show that $V^{P_{\omega_2}} \models$ "$(\forall G \in \mathcal{K}_i)$ *G is not quasi-smooth*" for $i < 3$. Take

$$\mathcal{K}_0 = \{G \in \mathcal{K}\colon \exists A \in [\omega_1]^\omega\ |G/\equiv_{G,A}| = \omega_1\},$$

$$\mathcal{K}_1 = \{G \in \mathcal{K}\colon \forall A \in [\omega_1]^\omega\ \exists x\ |\omega_1 \setminus [x]_{G,A}| < \omega_1\}$$

and

$$\mathcal{K}_2 = \mathcal{K} \setminus (\mathcal{K}_0 \cup \mathcal{K}_1).$$

4.1. $G \in \mathcal{K}_0$. First we recall a definition of [1].

Definition 4.2: A poset P is **stable** if

$$\forall B \in [P]^\omega\ \exists B^* \in [P]^\omega\ \forall p \in P\ \exists p' \leq p\ \exists p^* \in B^*\ \forall b \in B\ (p' \|_P b \text{ iff } p^* \|_P b).$$

We will say that p' and p^* are **twins for** B and that B^* **shows the stability of** P **for** B. ∎

Lemma 4.3: *P_{ω_2} is stable.*

Proof: First let us remark that it is enough to prove that both $\mathcal{C}$ and $\mathcal{Q}_T$ are stable for any Aronszajn-tree for in [1] it was proved that any finite support iteration of stable, c.c.c. posets is stable.

It is clear that $\mathcal{C}$ is stable. Assume that T is an Aronszajn tree and $B \subset [\mathcal{Q}_T]^\omega$. Fix a countable ordinal δ with $\{\mathrm{dom}(p)\colon p \in B\} \subset T_{<\delta}$ and take $B^* = \{p \in \mathcal{Q}_T\colon \mathrm{dom}(p) \subset T_{<\delta+\omega}\}$. It is not hard to see that B^* shows the stability of P for B. ∎

For $G \in \mathcal{K}$ take $G \in \mathcal{K}_0^*$ iff there is an $A \in [\omega_1]^\omega$ such that the set $\{x\colon |[x]_{G,A}| \le \omega\}$ is uncountable.

Given $G \in \mathcal{K}_0$ we will write $G \in \mathcal{K}_0'$ iff there are disjoint sets $A_0, A_1 \in [\omega_1]^\omega$ such that

(1) $x \equiv_{G,A_0} y$ iff $x \equiv_{G,A_1} y$ for each $x, y \in \omega_1 \setminus A_0 \cup A_1$,

(2) the set $\{x\colon |[x]_{G,A_0}| \le \omega\}$ is uncountable.

Lemma 4.4: *Assume CH. If $G \in \mathcal{K}_0'$, then there is a partition (V_0, V_1) of ω_1 so that for each stable c.c.c. poset P we have*

$$V^P \models \text{"}G \text{ is not isomorphic to } G[V_i] \text{ for } i \in 2\text{"}.$$

Proof: Pick $A_0, A_1 \in [\omega_1]^\omega$ witnessing $G \in \mathcal{K}_0'$. Write $A = A_0 \cup A_1$. Take $E = E(G)$.

Let κ be a large enough regular cardinal and fix an increasing sequence $\langle N_\nu : \nu < \omega_1\rangle$ of countable, elementary submodels of $\mathcal{H}_\kappa$ such that

(i) $G, A, A_0, A_1 \in N_0$,

(ii) $\langle N_\nu : \nu < \mu\rangle \in N_\mu$ for $\mu < \omega_1$.

For $x \in \omega_1 \setminus A$ take

$$\mathrm{rank}(x) = \min\{\nu\colon x \in N_\nu\}.$$

Fix a partition (S_0, S_1) of ω_1 with $|S_0| = |S_1| = \omega_1$. Take

$$V_i = A_i \cup \mathrm{rank}^{-1} S_i$$

for $i \in 2$.

We show that the partition (V_0, V_1) works.

Assume on the contrary that P is a stable c.c.c. poset, $\dot f$ is a P-name of a function, $p_0 \in P$ and

$$p_0 \Vdash \text{"}\dot f\colon G \cong G[V_0]\text{"}.$$

Without loss of generality we can assume that $p_0 = 1_P$. Now for each $c \in A_0$ choose a maximal antichain $J_c \subset P$ and a function $h_c\colon J_c \to V$ such that $q \Vdash \text{"}\dot f^{-1}(\hat c) = \widehat{h_c(r)}\text{"}$ for each $q \in J_c$.

Take $B = \bigcup\{J_c\colon c \in A_0\}$ and pick a countable $B^* \subset P$ showing the stability of P for B.

For $b \in P$ define the partial function $\mathrm{dt}_b\colon \omega_1 \to 2^{A_0}$ as follows. Let $x \in \omega_1$. If there is a function $t \in 2^{A_0}$ so that

(a) $t(c) = 1 \iff$ for each $q \in I_c$ if q and b are compatible conditions, then $\{x, h_c(q)\} \in E$,

(b) $t(c) = 0 \iff$ for each $q \in I_c$ if q and b are compatible conditions, then $\{x, h_c(q)\} \notin E$,

then take $\mathrm{dt}_b(x) = t$. Otherwise $x \notin \mathrm{dom}\,\mathrm{dt}_b$.

SUBLEMMA 4.4.1: *If $p \Vdash$ "$\dot{f}(x) = y$", then there are $p' \leq p$ and $b \in B^*$ such that b and p' are twins for B and $\mathrm{dt}_b(x) = \mathrm{tp}_G(y, A_0)$.*

Proof: By the choice of B^*, we can find a $p' \leq p$ and a $b \in B^*$ so that p' and b are twins for B. Let $c \in A_0$. For each $q \in J_c$, if q and p' are compatible in P, then $\{y, c\} \in E$ iff $\{x, h_c(q)\} \in E$, because, taking r as a common extension of q and p', we have $r \Vdash$ "$\dot{f}(\hat{x}) = \hat{y}$ *and* $\dot{f}(\widehat{h_q(c)}) = \hat{c}$". So $\{y, c\} \in E$ iff for each $q \in I_c$ if q and p' are compatible, then $\{x, h_c(q)\} \in E$. But p' and b are twins for $\bigcup\{J_c\colon c \in A_0\}$, so $\mathrm{dt}_b(x) = \mathrm{dt}_p(x) = \mathrm{tp}_G(y, A_0)$. ∎

SUBLEMMA 4.4.2: *There is a $b \in B^*$ such that*

$$(*) \qquad |\{t \in \mathrm{ran}\,\mathrm{dt}_b\colon |rl^*_G(t)| \leq \omega\}| = \omega_1.$$

Proof: Let $\mathcal{G}$ be a P-generic filter over V. Put

$$\mathcal{F} = \{\mathrm{tp}_G(y, A_0)\colon y \in V_0 \setminus A_0, |[y]_{G,A_0}| \leq \omega\}.$$

Then $|\mathcal{F}| = \omega_1$, so we can write $\mathcal{F} = \{t_\nu : \nu < \omega_1\}$. Fix sequences $\langle p_\nu : \nu < \omega_1\rangle \subset \mathcal{G}$, $\langle x_\nu : \nu < \omega_1\rangle \subset \omega_1$ and $\langle y_\nu : \nu < \omega_1\rangle \subset \omega_1$ such that $p_\nu \Vdash$ "$\dot{f}(x_\nu) = y_\nu$" and $\mathrm{tp}_G(y_\nu, A_0) = t_\nu$. By Sublemma 4.4.1,

$$\bigcup_{b \in B^*} \mathrm{ran}\,\mathrm{dt}_b \supseteq \mathcal{F}.$$

But B^* is countable, so we can find a $b \in B^*$ satisfying $(*)$ above. ∎

Fix $b \in B^*$ with property $(*)$. Consider the structure

$$\mathcal{N} = \langle P\lceil(B \cup B^*), B, B^*, \langle J_c, h_c\colon c \in A_0\rangle\rangle.$$

By CH, there is a $\nu < \omega_1$ with $\mathcal{N} \in N_\nu$. Pick $\mu \in S_1 \setminus \nu$. Since $G, \mathcal{N}, b \in N_\nu$, it follows that $\mathrm{dt}_b \in N_\nu \subset N_\mu$. By (∗) and (ii), there is a

$$t \in \operatorname{ran} \mathrm{dt}_b \cap (N_\mu \setminus \bigcup_{\xi<\mu} N_\xi)$$

with $|\mathrm{rl}^*_G(t)| \le \omega$. Then

$$(\dagger) \qquad \mathrm{rl}^*_G(t) \subset N_\mu \setminus \bigcup_{\xi<\mu} N_\xi.$$

Pick $x \in \omega_1$ with $\mathrm{dt}_b(x) = t$. Find $p \in \mathcal{G}$ and $y \in V_0$ such that $p \le b$ and $p \Vdash$ "$\dot{f}(x) = y$". By Sublemma 4.4.1, there are $p' \le p$ and $b' \in B^*$ such that p' and b' are twins for B and $\mathrm{dt}_{b'}(x) = \mathrm{tp}_G(y, A_0)$. But $p \le b$, so $\mathrm{dt}_b(x) = \mathrm{dt}_{b'}(x)$. Indeed, let $c \in A_0$ and assume that $\mathrm{dt}_{b'}(x)(c) = 1$. Pick $q \in J_q$ which is compatible with b'. By the definition of $\mathrm{dt}_{b'}$, it follows that $\{h_c(q), x\} \in E$. Since p' and b' are twins for B, so p' and q also have a common extension q' in P. But $p' \le p \le b$, so q' witnesses that b and q are compatible. Thus, by the definition of dt_b, we have $\mathrm{dt}_b(x)(c) = 1$.

Thus $\mathrm{tp}_G(y, A_0) = \mathrm{dt}_{b'}(x) = \mathrm{dt}_b(x) = t$. By (†), this implies that $\mathrm{rank}(y) = \mu$. But, by the construction of the partition (V_0, V_1), there are no $y \in V_0$ with $\mathrm{rank}(y) = \mu$. Contradiction, the lemma is proved. ∎

LEMMA 4.5: *Assume CH. If $G \in \mathcal{K}^*_0$, then $V^{\mathcal{C}} \models$ " there is a partition (V_0, V_1) of ω_1 so that for each stable c.c.c. poset P we have:*

$$V^{\mathcal{C}*P} \models \text{"}G \text{ is not isomorphic to } G[V_i] \text{ for } i \in 2\text{". "}$$

Proof: Fix a set $A \in [\omega_1]^\omega$ witnessing $G \in \mathcal{K}^*_0$ and a bijection $f: A \to \omega$ in V. Let $r: \omega \to 2$ be the characteristic function of a Cohen real from $V^{\mathcal{C}}$. Take $A_i = (f \circ r)^{-1}\{i\}$ for $i < 2$. Then (A_0, A_1) is a partition of A. Using a trivial density argument we can see that $x \not\equiv_{G,A} y$ implies $x \not\equiv_{G,A_i} y$ for $i < 2$ and for $x, y \in \omega_1 \setminus A$. Thus $V^{\mathcal{C}} \models$ "A_0 and A_1 witness $G \in \mathcal{K}'_0$". Applying Lemma 4.4 in $V^{\mathcal{C}}$ we get the desired partition of ω_1. ∎

LEMMA 4.6: *In $V^{P_{\omega_2}}$, if $G \in \mathcal{K}_0$ is quasi-smooth, then $G \in \mathcal{K}^*_0$.*

Proof: Choose a set $A \in [\omega_1]^\omega$ witnessing $G \in \mathcal{K}_0$ and a bijection $f: A \to \omega$. Pick $\alpha < \omega_2$, α is even, with A, f, $G \in V^{P_\alpha}$. From now on we work in V^{P_α}. Let $\{[x_\nu]_{G,A}: \nu < \omega_1\}$ be an enumeration of the equivalence classes of $\equiv_{G,A}$. Fix a partition (I_0, I_1) of ω_1 into uncountable pieces. Let $r\colon \omega \to 2$ be the characteristic function of a Cohen real from $V^{P_\alpha * \mathcal{C}}$. Take $A_i = (f \circ r)^{-1}\{i\}$ for $i < 2$. Then (A_0, A_1) is a partition of A. Using a trivial density argument we can see that $x \not\equiv_{G,A} y$ implies $x \not\equiv_{G,A_i} y$ for $i < 2$ and for $x, y \in \omega_1 \setminus A$. For $i \in 2$ put

$$B_i = A_i \cup \{x_\nu\colon \nu \in I_i\} \cup \{[x_\nu]_{G,A} \setminus \{x_\nu\}\colon \nu \in I_{1-i}\}.$$

Clearly (B_0, B_1) is a partition of ω_1 and

$$B_i \cap [x_\nu]_{G,A_i} = B_i \cap [x_\nu]_{G,A} = \{x_\nu\}.$$

So $G[B_i] \in \mathcal{K}_0^*$. But G is quasi-smooth, so $G \cong G[B_i]$ for some $i \in 2$ in $V^{P_{\omega_2}}$. Thus $G \in \mathcal{K}_0^*$ is proved. ∎

4.2. $G \in \mathcal{K}_1$. We say that a poset P has property *Pr* iff for each sequence $\langle p_\nu : \nu < \omega_1\rangle \subset P$ there exist disjoint sets $U_0, U_1 \in [\omega_1]^{\omega_1}$ such that whenever $\alpha \in U_0$ and $\beta \in U_1$ we have $p_\alpha \|_P p_\beta$.

LEMMA 4.7: *$\mathcal{C}$ has property Pr.*

Indeed, $\mathcal{C}$ has property K.

LEMMA 4.8: *If T is an Aronszajn-tree, then $\mathcal{Q}_T$ has property Pr.*

Proof: Let $\langle p_\alpha : \alpha < \omega_1\rangle \subset P$ be given. We can assume that there are a stationary set $S \subset \omega_1$, $p^* \in \mathcal{Q}_T$, $\gamma^* < \omega_1$, $n \in \omega$ and $\{z_i\colon i < n\} \subset T$ such that for each $\alpha \in S$

(a) $x\lceil\alpha \in \operatorname{dom}(p_\alpha)$ for each $x \in \operatorname{dom}(p_\alpha)$ with $\operatorname{height}_T(x) \geq \alpha$,
(b) $p_\alpha\lceil T_{<\alpha} = p^*$,
(c) $|\operatorname{dom}(p_\alpha) \cap T_\alpha| = n$,
(d) writing $\operatorname{dom}(p_\alpha) \cap T_\alpha = \{x_0^\alpha, \ldots, x_{n-1}^\alpha\}$, $x_0^\alpha <_{\mathrm{On}} \cdots <_{\mathrm{On}} x_{n-1}^\alpha$, the sequence $\langle p_\alpha(x_0^\alpha), \ldots, p_\alpha(x_{n-1}^\alpha)\rangle$ is independent from α,
(e) $\gamma^* < \alpha$ and the elements $x_0^\alpha\lceil\gamma^*, \ldots, x_{n-1}^\alpha\lceil\gamma^*$ are pairwise distinct,
(f) $x_i^\alpha\lceil\gamma^* = z_i$ for $i < n$.

For each $\beta < \omega_1$ and $\bar{y} = \langle y_0, \ldots, y_{n-1}\rangle \in (T_\beta)^n$ take

$$S_{\bar{y}} = \{\alpha \in S \setminus \beta\colon x_i^\alpha\lceil\beta = y_i \text{ for each } i < n\}.$$

Let

$$C^* = \{\delta < \omega_1 \setminus \gamma^*\colon \forall\beta < \delta\ \forall\bar{y} \in (T_\beta)^n\ (|S_{\bar{y}}| \leq \omega \to S_{\bar{y}} \subset \delta)\}.$$

Now $\{p_\alpha: \alpha \in S \cap C^*\}$ are ω_1 members of P, so for some $\alpha < \beta \in S \cap C^*$ the conditions p_α and p_β are compatible. Since $p_\alpha(x_l^\alpha) = p_\beta(x_l^\beta)$, x_l^α and x_l^β are incomparable in T for $l < n$. So for some $\nu < \alpha$, $x_l^\alpha \lceil \nu \neq x_l^\beta \lceil \nu$ whenever $l < n$. On the other hand, for $l \neq m < n$ we have $x_l^\alpha \lceil \nu \neq x_m^\beta \lceil \nu$ because $x_l^\alpha \lceil \gamma^* = z_l \neq z_m = x_m^\beta \lceil \gamma^*$. Take $y_l^a = x_l^\alpha \lceil \nu$ and $y_l^b = x_l^\beta \lceil \nu$ for $l < n$ and write $\bar{a} = \langle y_0^a, \dots, y_{n-1}^a \rangle$, $\bar{b} = \langle y_0^b, \dots, y_{n-1}^b \rangle$. The elements $\{y_i^a, y_i^b: i < n\}$ are pairwise different, so for each $\alpha' \in S_{\bar{a}}$ and $\beta' \in S_{\bar{b}}$ the conditions $p_{\alpha'}$ and $p_{\beta'}$ are compatible. But $|S_{\bar{a}}| = |S_{\bar{b}}| = \omega_1$, because $\alpha \in S_{\bar{a}}$, $\beta \in S_{\bar{b}}$, $\nu < \alpha$ and $\alpha \in C^*$. ∎

A poset P is called **well-met** if any two compatible elements p_0 and p_1 of P have a greatest lower bound denoted by $p_0 \wedge p_1$.

LEMMA 4.9: *Assume that the poset P has property Pr and $V^P \models$ "the poset Q has property Pr". Let $\{\langle p_\alpha, q_\alpha \rangle : \alpha < \omega_1\} \subset P * Q$. Then there are disjoint sets $U_0, U_1 \in [\omega_1]^{\omega_1}$ such that for each $\gamma \in U_0$ and $\delta \in U_1$ the conditions $\langle p_\gamma, q_\gamma \rangle$ and $\langle p_\delta, q_\delta \rangle$ are compatible, in other words, p_γ and p_δ have a common extension $p_{\gamma,\delta}$ in P with $p_{\gamma,\delta} \Vdash$ "$q_\gamma \parallel_Q q_\delta$". If P is well-met, then we can find conditions $\{p'_\alpha: \alpha \in U_0 \cup U_1\}$ in P with $p'_\alpha \le p_\alpha$ such that $p'_\gamma \wedge p'_\delta \Vdash$ "$q_\gamma \parallel_Q q_\delta$" for each $\gamma \in U_0$ and $\delta \in U_1$.*

Proof: Let $\dot{U}$ be a P-name for the set $U = \{\alpha: p_\alpha \in \mathcal{G}_P\}$, where $\mathcal{G}_P$ is the P-generic filter. Since P satisfies c.c.c., there is a $p^* \in P$ with $p^* \Vdash$ "$|\dot{U}| = \omega_1$". Since $V^P \models$ "*Q has property Pr*", there is a condition $p \le p^*$ and there are P-names such that $p \Vdash$ "*$V_i = \{\dot{\alpha}_\gamma^i: \gamma < \omega_1\} \in [U]^{\omega_1}$, for $i \in 2$, and $q_{\dot{\alpha}_\gamma^0}$ and $q_{\dot{\alpha}_\delta^1}$ are compatible whenever $\gamma, \delta \in \omega_1$*". Choose conditions $p_\gamma^* \le p$ and ordinals β_γ^0, β_γ^1, with $p_\gamma^* \Vdash$ " $\dot{\alpha}_\gamma^i = \hat{\beta}_\gamma^i$" for $i < 2$.

Now consider the sequence $A = \{p_\gamma^*: \gamma < \omega_1\}$. Since P has property Pr, there are disjoint, uncountable sets $C_0, C_1 \subset A$ such that p_γ^* and p_δ^* are compatible whenever $\gamma \in C_0$ and $\delta \in C_1$. Take $U_i = \{\beta_\gamma^i: \gamma \in C_i\}$ for $i \in 2$. We can assume that $U_0 \cap U_1 = \emptyset$. Let $\gamma \in C_0$ and $\delta \in C_1$ and let p'' be a common extension of p_γ^* and p_δ^*. Then $p'' \Vdash$ "$\beta_\gamma^0, \beta_\delta^1 \in \dot{U}$, *that is, $p_{\beta_\gamma^0}$ and $p_{\beta_\delta^1}$ are in $\mathcal{G}_P$*", so p'', must be a common extension of $p_{\beta_\gamma^0}$ and $p_{\beta_\delta^1}$. So $p'' \Vdash$ "$\beta_\gamma^0 \in V_0$ and $\beta_\delta^1 \in V_1$", thus $p'' \Vdash$ "$q_{\beta_\gamma^0}$ and $q_{\beta_\delta^1}$ are compatible in Q", so $\left\langle p_{\beta_\gamma^0}, q_{\beta_\gamma^0} \right\rangle \parallel_{P*Q} \left\langle p_{\beta_\delta^1}, q_{\beta_\delta^1} \right\rangle$.

Suppose that P is well-met. Take $p'_{\beta_\gamma^0} = p_\gamma^* \wedge p_{\beta_\gamma^0}$ and $p'_{\beta_\delta^1} = p_\delta^* \wedge p_{\beta_\delta^1}$. It works because we can use $p_\gamma^* \wedge p_\delta^*$ as p'' in the argument of the previous paragraph. ∎

LEMMA 4.10: *If $\langle R_\alpha: \alpha \le \mu, S_\beta: \beta < \mu\rangle$ is a finite support iteration such that $V^{R_\alpha} \models$ "S_α has property Pr" for $\alpha < \mu$, then R_μ has property Pr, as well.*

Proof: We prove this lemma by induction on μ. The successor case is covered by lemma 4.9. Assume that μ is limit. Let $\langle p_\xi : \xi < \omega_1\rangle \subset R_\mu$. Without loss of generality we can assume that $\langle \mathrm{supp}(p_\xi): \xi < \omega_1\rangle$ forms a Δ-system with kernel d. Fix $\nu < \mu$ with $d \subset \nu$. By the induction hypothesis, the poset R_ν has property *Pr*, so there exist disjoint sets U_0, $U_1 \in [\omega_1]^{\omega_1}$ such that whenever $\xi \in U_0$ and $\eta \in U_1$ we have $p_\xi \|_{R_\nu} p_\eta$. But $p_\xi \|_{R_\nu} p_\eta$ implies $p_\xi \|_{R_\mu} p_\eta$ because $\mathrm{supp}(p_\xi) \cap \mathrm{supp}(p_\eta) \subset \nu$, so R_μ has property *Pr*, as well. ∎

The previous lemmas yield the following corollary.

LEMMA 4.11: *P_{ω_2} has property Pr.*

Given $G = \langle \omega_1, E\rangle \in \mathcal{K}_1$ and ξ, α, $\beta \in \omega_1$ with $\xi \in \alpha \cap \beta$ take

$$\mathrm{D}^G_\xi(\alpha,\beta) = \{\nu \in \xi: \{\alpha,\nu\} \in E \text{ iff } \{\beta,\nu\} \notin E\}.$$

LEMMA 4.12: *If $G \in \mathcal{K}_1$, then*

$$(*) \qquad \forall \xi \in \omega_1\ \exists \epsilon_G(\xi) \in \omega_1\ \forall \alpha,\beta \in \omega_1 \setminus \epsilon_G(\xi)\ |\mathrm{D}^G_\xi(\alpha,\beta)| < \omega.$$

Proof: Since $G \in \mathcal{K}_1$, we have an $x \in \omega_1$ with $|\omega_1 \setminus [x]_\xi| < \omega_1$. Choose $\epsilon_G(\xi) \in \omega_1 \setminus \xi$ with $\omega_1 \setminus [x]_\xi \subset \epsilon_G(\xi)$. It works because α, $\beta > \epsilon_G(\xi)$ implies α, $\beta \in [x]_\xi$. ∎

The bipartite graph $\langle \omega_1 \times 2, \{\{\langle \nu, 0\rangle, \langle \mu, 1\rangle\}: \nu < \mu < \omega_1\}\rangle$ will be denoted by $[\omega_1;\omega_1]$.

LEMMA 4.13: *If $G \in \mathcal{K}_1$, then neither G nor its complement may have a — not necessarily spanned — subgraph isomorphic to $[\omega_1;\omega_1]$.*

Proof: Let $G = \langle \omega_1, E\rangle$. Write $E(\alpha) = \{\xi \in \omega_1: \{\xi,\alpha\} \in E\}$. Assume on the contrary that A, $B \in [\omega_1]^{\omega_1}$ are disjoint sets such that $\{\alpha,\beta\} \in E$ whenever $\alpha \in A$ and $\beta \in B$ with $\alpha < \beta$. Without loss of generality we can assume that $(A \setminus \alpha + 1) \cap \epsilon(\alpha) = \emptyset$ for each $\alpha \in A$. Write $A = \{\alpha_\xi : \xi < \omega_1\}$. Then for $\xi \in \omega_1$ the set $F(\xi) = (A \cap \alpha_\xi) \setminus E(\alpha_{\xi+1})$ is finite because $\alpha_{\xi+1} > \epsilon(\alpha_\xi)$ and $(A \cap \alpha_\xi) \setminus E(\beta) = \emptyset$ for all but countable many $\beta \in B$. By Fodor's lemma, we can assume that $F(\xi) = F$ for each $\xi \in S$, where S is a stationary subset of ω_1 containing limit ordinals only. Let $T = \{\xi \in S: F \subset \alpha_\xi\}$ and take $W = \{\alpha_{\xi+1}: \xi \in T\}$. Then $G[W]$ is an uncountable complete subgraph of G. Contradiction. ∎

LEMMA 4.14: *If $G \in \mathcal{K}_1$ and $V^{\mathcal{C}} \models$ "Q has property Pr", then*

$$V^{\mathcal{C}*Q} \models \text{“}G \not\cong G[f^{-1}\{i\}] \text{ for } i < 2\text{”},$$

where $f\colon \omega_1 \to 2$ is the $\mathcal{C}$-generic function over V.

Proof: Assume on the contrary that

$$\langle p, q\rangle \Vdash \text{“}\dot{h}\colon G \cong G[f^{-1}\{0\}]\text{”}.$$

To simplify our notations, we will write E for $E(G)$, $\mathrm{D}_\xi(\alpha,\beta)$ for $\mathrm{D}^G_\xi(\alpha,\beta)$ and $\epsilon(\xi)$ for $\epsilon_G(\xi)$.

Let $C_0 = \{\delta < \omega_1\colon \xi < \delta$ implies $\epsilon(\xi) < \delta\}$. Clearly C_0 is club. Take $C_1 = \{\delta < \omega_1\colon \langle p, q\rangle \Vdash \text{“}\dot{h}''\hat{\delta} = f^{-1}\{0\} \cap \hat{\delta}\text{”}\}$. Since $\mathcal{C} * Q$ satisfies c.c.c, the set C_1 is club. Put $C_2 = C_0 \cap C_1$.

Now for each $\alpha < \omega_1$ let $\delta_\alpha = \min(C_2 \setminus \alpha{+}1)$ and choose a condition $\langle p_\alpha, q_\alpha\rangle \leq \langle p, q\rangle$ and a countable ordinal γ_α such that

$$\langle p_\alpha, q_\alpha\rangle \Vdash \text{“}\dot{h}(\hat{\delta}_\alpha) = \hat{\gamma}_\alpha\text{”}.$$

Since $\gamma_\alpha \geq \delta_\alpha > \epsilon(\alpha)$ for each $\alpha \in \omega_1$, we can fix a stationary set $S \subset \omega_1$ and a finite set D such that $\mathrm{D}_\alpha(\delta_\alpha, \gamma_\alpha) = \mathrm{D}$ for each $\alpha \in S$. Since $\mathcal{C}$ is well-met, applying lemma 4.9 we can find disjoint uncountable subsets $S_0, S_1 \subset S$ and a sequence $\langle p'_\alpha\colon \alpha \in S_0 \cup S_1\rangle \subset \mathcal{C}$ with $p'_\alpha \leq p_\alpha$ such that $p'_\alpha \wedge p'_\beta \Vdash$ "$q_\alpha \parallel_Q q_\beta$" for each $\alpha \in S_0$ and $\beta \in S_1$.

We can assume that the sets $\{\mathrm{dom}(p'_\alpha)\colon \alpha \in S_0\}$ and $\{\mathrm{dom}(p'_\beta)\colon \beta \in S_1\}$ form Δ-systems with kernels d_0 and d_1, respectively.

Take $Y^0_\xi = \{\alpha \in S_0\colon \{\xi, \delta_\alpha\} \in E\}$ and $Y^1_\xi = \{\alpha \in S_1\colon \{\xi, \delta_\alpha\} \notin E\}$ for $\xi < \omega_1$. Write $Y_i = \{\xi < \omega_1\colon |Y^i_\xi| = \omega_1\}$ and $Z_i = \omega_1 \setminus Y_i$ for $i < 2$.

By 4.13, the sets Z_i are countable. Pick $\xi \in C_2$ with $D \cup d_0 \cup d_1 \cup Z_0 \cup Z_1 \subset \xi$. Let $\xi' = \min(C_2 \setminus \xi + 1)$ and $\xi'' = \min(C_2 \setminus \xi' + 1)$. Since $d_0 \cup d_1 \subset \xi$ and $|Y^0_\xi| = |Y^1_\xi| = \omega_1$, we can choose $\alpha_i \in Y^i_\xi \setminus \xi''$ with $\mathrm{dom}(p'_{\alpha_i}) \cap [\xi, \xi') = \emptyset$ for $i = 0, 1$. The set $W = \mathrm{D}_{\xi'}(\delta_{\alpha_0}, \delta_{\alpha_1}) \cap [\xi, \xi')$is finite because $\delta_{\alpha_i} \geq \alpha_i \geq \xi'' > \epsilon(\xi')$ for $i < 2$. Choose a $\mathcal{C}$-name q such that $p'_{\alpha_0} \wedge p'_{\alpha_1} \Vdash$ "*q is a common extension of q_{α_0} and q_{α_1} in Q*" and take

$$r = \left\langle p'_{\alpha_0} \cup p'_{\alpha_1} \cup \{\langle \nu, 1\rangle : \nu \in W\}, q\right\rangle.$$

Since $W \cap (\mathrm{dom}(p'_{\alpha_0}) \cup \mathrm{dom}(p'_{\alpha_1})) = \emptyset$, r is a condition.

Pick a condition $r' \leq r$ from $\mathcal{C} * Q$ and an ordinal η such that $r' \Vdash$"$\dot{h}(\hat{\xi}) = \hat{\eta}$". Now $\eta \in [\xi, \xi')$ because ξ, $\xi' \in C_1$. Since

$$r' \Vdash \text{"}\dot{h}(\delta_{\alpha_i}) = \hat{\gamma}_{\alpha_i}, \dot{h}(\hat{\xi}) = \hat{\eta} \text{ and } \dot{h} \text{ is an isomorphism"},$$

so $\{\delta_{\alpha_0}, \xi\} \in E$ and $\{\delta_{\alpha_1}, \xi\} \notin E$ imply that $\{\gamma_{\alpha_0}, \eta\} \in E$ and $\{\gamma_{\alpha_1}, \eta\} \notin E$. But $D_{\alpha_i}(\delta_{\alpha_i}, \gamma_{\alpha_i}) = D$ and $D \subset \xi$ so $\{\delta_{\alpha_0}, \eta\} \in E$ and $\{\delta_{\alpha_1}, \eta\} \notin E$, that is, $\eta \in W$. But $r \Vdash$"$\mathrm{ran}(\dot{h}) = f^{-1}\{0\}$ *and* $f^{-1}\{0\} \cap \hat{W} = \emptyset$", contradiction. ∎

4.3 $G \in \mathcal{K}_2$. Given a non-trivial graph $G = \langle V, E\rangle$ with $V \in [\omega_1]^{\omega_1}$ define

$$\Gamma(G) = \{\delta \in \omega_1 \colon \exists \alpha \in V\ \alpha \geq \delta \text{ and } |\mathrm{twin}_G(\alpha, V \cap \delta)| \leq \omega\}.$$

The following lemma obviously holds.

Lemma 4.15: *If G_0 and G_1 are graphs on uncountable subsets of ω_1, $G_0 \cong G_1$, then* $\Gamma(G_0) = \Gamma(G_1) \bmod \mathrm{NS}_{\omega_1}$.

Lemma 4.16: *Given $G \in \mathcal{K} \setminus \mathcal{K}_0$ and $S \subset \omega_1$ there is a partition (V_0, V_1) of ω_1 such that $\Gamma(G[V_0]) \subset S \bmod \mathrm{NS}_{\omega_1}$ and $\Gamma(G[V_1]) \subset \omega_1 \setminus S \bmod \mathrm{NS}_{\omega_1}$.*

Proof: Let κ be a large enough regular cardinal and fix an increasing, continuous sequence $\langle N_\nu : \nu < \omega_1\rangle$ of countable, elementary submodels of $\mathcal{H}_\kappa = \langle H_\kappa, \in\rangle$ such that $G, S \in N_0$ and $\langle N_\nu \colon \nu \leq \mu\rangle \in N_{\mu+1}$ for $\mu < \omega_1$. Write $\gamma_\nu = N_\nu \cap \omega_1$ and $C = \{\gamma_\nu : \nu < \omega_1\}$. Take

$$V_0 = \bigcup_{\nu \in S} (\gamma_{\nu+1} \setminus \gamma_\nu) \quad \text{and} \quad V_1 = \omega_1 \setminus V_0 = \bigcup_{\nu \in \omega_1 \setminus S} (\gamma_{\nu+1} \setminus \gamma_\nu).$$

It is enough to prove that $\Gamma(G[V_0]) \subset S \bmod \mathrm{NS}_{\omega_1}$. Assume that $\gamma_\nu \in \Gamma(G[V_0])$, $\gamma_\nu = \nu$, $\alpha \geq \gamma_\nu$, $\alpha \in V_0$ and $|\mathrm{twin}_{G[V_0]}(\alpha, \gamma_\nu \cap V_0)| = \omega$. Since G, ν, $\gamma_\nu \cap V_0 \in N_{\nu+1}$ and $|G/\equiv_{G, V_0 \cap \gamma_\nu}| \leq \omega$, we have $\mathrm{tp}_{GG[V_0]}(\alpha, \gamma_\nu \cap V_0) \in N_{\nu+1}$ and so $\mathrm{twin}_{G[V_0]}(\alpha, \gamma_\nu) \subset N_{\nu+1}$ as well. Thus $\alpha \in \gamma_{\nu+1} \setminus \gamma_\nu$. Hence $\alpha \in V_0$ implies $\gamma_n = \nu \in S$ which was to be proved. ∎

Lemma 4.17: *If $G \in \mathcal{K} \setminus \mathcal{K}_0$ and $\Gamma(G) \neq \emptyset \bmod \mathrm{NS}_{\omega_1}$, then G is not quasi-smooth.*

Proof: Assume that $S = \Gamma(G)$ is stationary and let (S_0, S_1) be a partition of S into stationary subsets. By Lemma 4.16, there is a partition (V_0, V_1) of ω_1 with

$\Gamma(G([V_i]) \cap S \subset S_i$. Then $G[V_i]$ and G can not be isomorphic by Lemma 4.15. ∎

Let us remark that $G \in \mathcal{K}_2$ iff $G \in \mathcal{K} \setminus \mathcal{K}_0$ and there is an $A \in [\omega_1]^\omega$ and $x \in \omega_1 \setminus A$ such that $|[x]_{G,A}| = |\omega_1 \setminus [x]_{G,A}| = \omega_1$.

Given $G \in \mathcal{K}_2$ we will write $G \in \mathcal{K}_2'$ iff there are two disjoint, countable subsets of ω_1, A_0 and A_1, and there is an $x \in \omega_1$, such that $|[x]_{G,A_0}| = |\omega_1 \setminus [x]_{G,A_0}| = \omega_1$ and $[x]_{G,A_0} \setminus A_1 = [x]_{G,A_1} \setminus A_0$.

LEMMA 4.18: *If $G \in \mathcal{K}_2$, then $G \in (\mathcal{K}_2')^{V^C}$.*

Proof: Assume that $A \in [\omega_1]^\omega$ and $x \in \omega_1$ witness $G \in \mathcal{K}_2$ in the ground model. Fix a bijection $f\colon A \to \omega$ in V. Let $r\colon \omega \to 2$ be the characteristic function of a Cohen real from V^C. Take $A_i = (f \circ r)^{-1}\{i\}$. By a simple density argument, we can see that $[x]_{G,A_0} = [x]_{G,A} = [x]_{G,A_1}$. Thus A_0, A_1 and x show that $g \in \mathcal{K}_2'$. ∎

LEMMA 4.19: *Assume that every Aronszajn tree is special. If $G \in \mathcal{K}_2'$, then there is a partition (V_0, V_1) of ω_1 such that $\Gamma(G[V_i])$ is stationary for $i < 2$.*

Proof: Choose A_0, A_1 and x witnessing $G \in \mathcal{K}_2'$. Let $A = A_0 \cup A_1$. Take $C_0 = [x]_{G,A_0} \setminus A$, $C_1 = (\omega_1 \setminus [x]_{G,A_0}) \setminus A$ and consider the partition trees $\mathcal{T}_i$ of $G[C_i]$ for $i \in 2$ (see Definition 2.3). These trees are Aronszajn-trees because G is non-trivial. Fix functions $h_i\colon C_i \to \omega$ specializing $\mathcal{T}_i$. We can find natural numbers n_0 and n_1 such that the sets $S_i = \{\nu\colon h_i^{-1}\{n_i\} \cap (\mathcal{T}_i)_\nu \neq \emptyset\}$ are stationary, that is, $h_i^{-1}\{n_i\}$ meets stationary many levels of $\mathcal{T}_i$. Take $B_i = h_i^{-1}\{n_i\}$ and $Y_i = \{c \in C_i\colon \exists b \in B_i\ c \preceq_{\mathcal{T}_i} b\}$.

Pick any $\delta \in S_i$. Let $b \in B_i \cap (\mathcal{T}_i)_\delta$. If $c \in Y_i \setminus (\mathcal{T}_i)_{<\delta}$, $c \neq b$, then $c\lceil\delta \neq b$ by the construction of Y_i. So $\mathrm{tp}_{GG[Y_i]}(c, (\mathcal{T}_i)_{<\delta}) = \mathrm{tp}_{GG[Y_i]}(c\lceil\delta, (\mathcal{T}_i)_{<\delta}) \neq \mathrm{tp}_{GG[Y_i]}(b, (\mathcal{T}_i)_{<\delta})$ by the definition of the partition tree. This means that

$$\mathrm{twin}_{G[Y_i]}(b, (\mathcal{T}_i)_{<\delta}) = \{b\}.$$

Thus $\delta \in \Gamma(G[Y_i])$ provided $(\mathcal{T}_i)_{<\delta} \subset \delta$ and $b \geq \delta$. But these requirements exclude only a non-stationary subset of S_i. So $\Gamma(G[Y_i]) \supset S_i \bmod \mathrm{NS}_{\omega_1}$.

Let $V_i = Y_i \cup A_i \cup (C_{1-i} \setminus Y_{1-i})$ for $i \in 2$ and consider the partition (V_0, V_1) of ω_1 . If $z \in V_i \setminus (Y_i \cup A_i)$, then $\mathrm{tp}_G(z, A_i) \neq \mathrm{tp}_G(b, A_i)$ for any $b \in B_i$ because $C_0 \subset [x]_{G,A_i}$ and $C_1 \subset \omega_1 \setminus [x]_{G,A_i}$. So $\Gamma(G[V_i]) \supset S_i \bmod \mathrm{NS}_{\omega_1}$ holds. ∎

Now we are ready to conclude the proof of Theorem 4.1. We will work in $V^{P_{\omega_2}}$. Assume that $G \in \mathcal{K}$. We must show that G is not quasi-smooth.

Pick a $\nu < \omega_2$ with $G \in (\mathcal{K})^{V^{P_\nu}}$ and $Q_\nu = \mathcal{C}$. Assume first that $G \in (\mathcal{K}_0)^{V^{P_\nu}}$. If G were quasi-smooth in $V^{P_{\omega_2}}$, $G \in (\mathcal{K}_0^*)^{P_{\omega_2}}$ would hold by Lemma 4.6. So we can assume that $G \in (\mathcal{K}_0^*)^{P_\nu}$. Since P_{ω_2} is a stable, c.c.c. poset, so is $P_{\omega_2}/P_{\nu+1}$. So, by Lemma 4.5, there is a partition (V_O, V_1) of ω_1 in $V^{P_{\nu+1}}$ such that $V^{P_{\omega_2}} \models$"*G is not isomorphic to $G[V_i]$ for $i < 2$*".

Assume that $G \in (\mathcal{K}_1)^{V^{P_\nu}}$. Since P_{ω_2} has property Pr, so is $P_{\omega_2}/P_{\nu+1}$. Thus, by Lemma 4.14, the partition (V_O, V_1) of ω_1 given by the Q_ν-generic Cohen reals in $V^{P_{\nu+1}}$ has the property that $V^{P_{\omega_2}} \models$"*G is not isomorphic to $G[V_i]$ for $i < 2$*".

Finally assume that $G \in (\mathcal{K}_2)^{V^{P_\nu}}$. By Lemma 4.18, we have $G \in (\mathcal{K}_2')^{V^{P_{\nu+1}}}$. Since P_{ω_2} satisfies c.c.c, it follows that $G \in (\mathcal{K}_2')^{V^{P_{\omega_2}}}$. So applying Lemma 4.19 we can find a partition (V_0, V_1) of ω_1 such that both $\Gamma(G[V_0])$ and $\Gamma(G[V_1])$ are stationary. Thus, by Lemma 4.17, neither $G[V_0]$ nor $G[V_1]$ are quasi-smooth. So G itself can not be quasi-smooth. ∎

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